Exact Tonal Analysis on Polynomials (ETAP) for Computational Harmonic Analysis

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Abstract:
A computational harmonic analysis technique, ETAP is developed from the first principle. A closed-form formula for harmonics addition is presented in this paper as the Harmonic Addition Theorem (HAT). Power of cosine formula is applied with mathematical pattern such as checker box triangle (CBT) to exactly compute the amplitude and phase of the harmonics at the output of a polynomial nonlinearity.

Index Terms: Computational Harmonic Analysis, Harmonic Addition Theorem, Polynomials, and Harmonic Distortions.

I. Introduction

Let us define a polynomial nonlinearity as

$$f(x)=\sum_{n=0}^{Q} h_n x^n, \tag{1}$$

where h_n, and Q denote the polynomial coefficient of the *n*th term, and the highest degree, respectively. Let us define the input single tone with arbitrary amplitude A and phase in φ radian as

$$x(t)=A\cos(\omega t+\varphi), \tag{2}$$

where ω is the angular frequency in radian per second and t is the time in seconds.

When the sinusoidal signal in (2) is applied to the polynomial nonlinearity in (1), the output can be represented as

$$y(t)=DC_0+\sum_{k=1} B_k\cos\left(k\omega t+\psi_k\right), \tag{3}$$

where DC_0 is the DC component; B_k and ψ_k are the amplitude and phase of the *k*th harmonic, respectively.

The research problem is defined as follows: given (1) and (2), to compute (3).

II. Harmonic Addition Theorem (HAT)

The HAT is the key ingredient to solve the problem.

Theorem: $\sum_{i=1}^{L} A_i\cos(\omega t+\varphi_i)=B\cos(\omega t+\psi)$, where

$$B=\sqrt{\sum_{i=1}^{L}A_i^2+2\sum_{i=1}^{L-1}\sum_{j=i+1}^{L}A_iA_j\cos(\varphi_i-\varphi_j)}, \tag{4}$$

$$\psi=\operatorname{atan2}\left(\frac{\sum_{i=1}^{L}A_i\sin\varphi_i}{\sum_{i=1}^{L}A_i\cos\varphi_i}\right), -\pi<\psi\le\pi. \tag{5}$$

Proof: Let $x_e(t)$ be denoted as a complex exponential function that is given by

$$x_e(t)=\sum_{i=1}^{L}A_i\exp(j\varphi_i)=B\exp(j\psi), \tag{6}$$

where B and ψ can be presented in terms of A_i and φ_i as shown in (4) and (5), respectively. For the computation of ψ, *atan2* function [1] is used to exactly locate the angle in any of the four quadrants in the complex plane. The ordinary *atan* function range is, however, $-\pi/2<\psi\le\pi/2$ in contrast to the *atan2* function range of $-\pi<\psi\le\pi$. For the both cases of positive and negative angles in (6), let us define

$$x_{e^-}(t)=\sum_{i=1}^{L}A_i\exp(-j\varphi_i)=B\exp(-j\psi), \tag{7}$$

$$x_{e^+}(t)=\sum_{i=1}^{L}A_i\exp(j\varphi_i)=B\exp(j\psi). \tag{8}$$

Using (7), (8), and Euler's formula,

$$\begin{aligned}
&\sum_{i=1}^{L}A_i\cos(\omega t+\varphi_i)\\
&=\frac{1}{2}\exp(j\omega t)\underbrace{\sum_{i=1}^{L}A_i\exp(j\varphi_i)}_{x_{e^+}(t)}+\frac{1}{2}\exp(-j\omega t)\underbrace{\sum_{i=1}^{L}A_i\exp(-j\varphi_i)}_{x_{e^-}(t)}\\
&=\frac{1}{2}\exp(j\omega t)\underbrace{B\exp(j\psi)}_{x_{e^+}(t)}+\frac{1}{2}\exp(-j\omega t)\underbrace{B\exp(-j\psi)}_{x_{e^-}(t)}\\
&=\frac{B}{2}\left[\exp\{j(\omega t+\psi)\}+\exp\{-j(\omega t+\psi)\}\right]\\
&=B\cos(\omega t+\psi). \qquad Q.E.D.
\end{aligned}$$

III. Exact Tonal Analysis on Polynomials (ETAP)

By Demoivre's formula, the following power of cosine trigonometric identity has been derived [2, 3].

$$\cos^n\theta=\begin{cases}\dfrac{2}{2^n}\displaystyle\sum_{j=0}^{\frac{n-1}{2}}\binom{n}{j}\cos\left[(n-2j)\theta\right] & (n=\text{odd}),\\[2ex] \dfrac{1}{2^n}\dbinom{n}{n/2}+\dfrac{2}{2^n}\displaystyle\sum_{j=0}^{\frac{n}{2}-1}\binom{n}{j}\cos\left[(n-2j)\theta\right] & (n=\text{even}).\end{cases} \tag{9}$$

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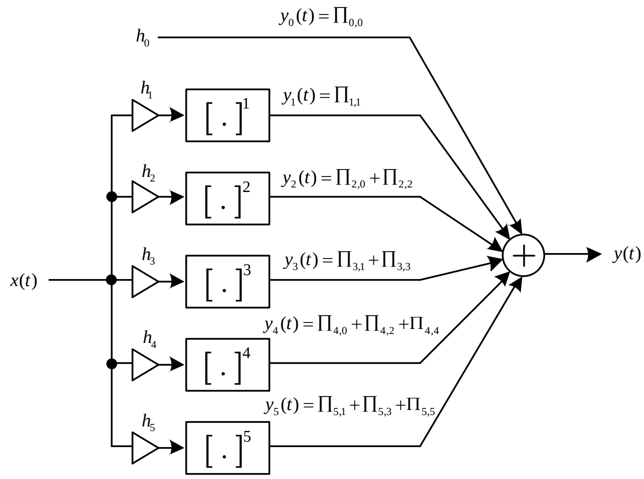

Fig. 1. Block diagram representation of (1) for the case of $Q=5$. Each branch represents (10), where their respective algebraic expansions are described in (11) and (12).

Let us denote the output signal at the nth degree polynomial branch in Fig. 1 as follows:

$$y_n(t)=h_n\left[x(t)\right]^n=h_n\left[A\cos\left(\omega t+\varphi\right)\right]^n$$
$$=\begin{cases}\sum_{k=1}^{n}\Pi_{n,k} & (n,k=\text{odd}),\\ \Pi_{n,0}+\sum_{k=2}^{n}\Pi_{n,k} & (n,k=\text{even}),\end{cases} \qquad (10)$$

where

$$\Pi_{n,k}=\begin{cases}A_{n,k}\cos\left(k\omega t+\varphi_{n,k}\right), & \text{for } k\neq 0,\\ DC_{n,0}, & \text{for } k=0,\end{cases} \qquad (11)$$

By (9),

$$A_{n,k}=2\left(A/2\right)^2 h_n\binom{n}{(n-k)/2},\ \text{and}\ \varphi_{n,k}=k\varphi \qquad (12)$$

are amplitude and phase of the kth harmonic at the nth term of the polynomial, respectively. By (1) and (10),

$$y(t)=\sum_{n=0}^{Q}y_n(t)=\sum_{n=0}^{Q}\sum_{\substack{k\in\{1,3,\ldots,n\},n=\text{odd}\\ k\in\{0,2,\ldots,n\},n=\text{even}}}\Pi_{n,k}, \qquad (13)$$

where $\Pi_{n,k}$ in (11) and (13) denotes the component generated at the n degree polynomial branch and k harmonic as illustrated in Fig. 1. When $\Pi_{n,k}$s are placed in the checker box, as shown in Fig. 2, the checker-box triangle (CBT) pattern is emerged. By (13),

$$y(t)=\sum_{n}\sum_{k}\Pi_{n,k}=\sum_{k}\sum_{n}\Pi_{n,k}. \qquad (14)$$

Thus, the component having the same frequency can be added together using (4) and (5). In pictorial representation (see Fig. 2), the components in CBT are added vertically using HAT. In symbolic representation,

$$\begin{aligned}y(t)&=\sum_{n=\text{even}}DC_{n,0}+\sum_{k>0}\sum_{n}A_{n,k}\cos\left(k\omega t+\varphi_{n,k}\right)\\&=DC_0+\sum_{k>0}B_k\cos\left(k\omega t+\psi_k\right).\end{aligned} \qquad (15)$$

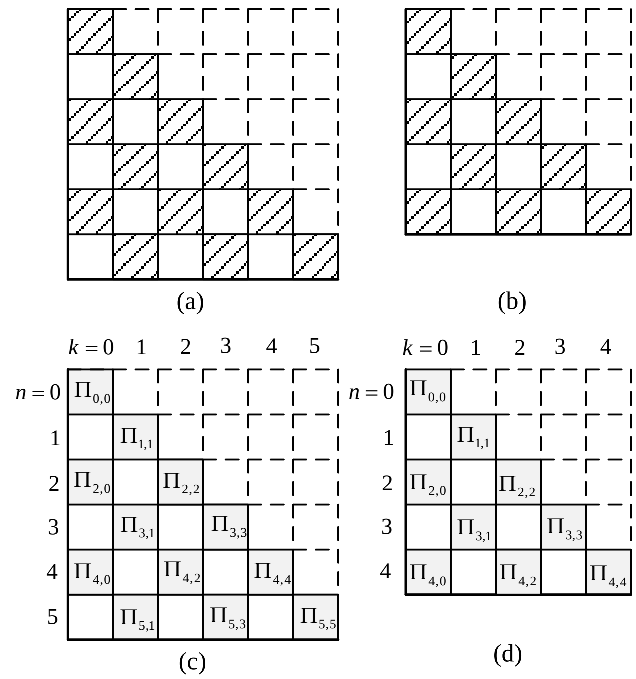

Fig. 2. Checker-box Triangle Pattern where each row and each column represent the component outputs from $y_n(t)$ and the generated harmonics with the exception of DC at $k=0$, respectively. Sub-figures (a), and (c) illustrates the generated component fill-ups for the case of $Q=5$, whereas (b), and (d) for $Q=4$. The indices n and k denote the index of the coefficient of polynomial, which is equivalent to the index of $y_n(t)$ and harmonic number respectively. Note that the component is DC when $k=0$.

As a computational example, let (2) with $A=1$ and $\varphi=\pi/2$ is applied into $f(x)=1.4214x-0.7409x^3+0.3313x^5$. The output signal in (3) or (15) is obtained as

$$\begin{aligned}y(t)&=1.7541\cos(\omega t+\pi/2)+0.2590\cos(3\omega t-\pi/2)\\&\quad+0.0888\cos(5\omega t+\pi/2).\end{aligned}$$

IV. Conclusion

A technique to compute the harmonic amplitudes, phases, and DC components at the output of polynomial nonlinearity was developed.

References

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